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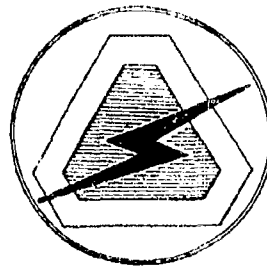


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ON THE THEORY OF THE ATMOSPHERIC ELECTRIC CURRENT FLOW, IV

Heinz W. Kasemir



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Heinz W. Kasemir

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Abstract

Starting from the differential equation of the continuity of the current flow, a general solution is given for problems in which the conductivity is given as a function of space and time. A physical interpretation for the complete Maxwell current is obtained, and it is shown how the Maxwell current can be composed of two field vectors, namely, that of the electrostatic field and of the stationary current flow. Each vector is weighted by a different time function, which can be calculated from the time function of the current source.

A method is developed for the calculation of the stationary current flow field; and the eigen functions for cartesian, cylindrical, and polar coordinates are given. The mirror law for the current flow is determined in a medium where the conductivity increases with altitude according to an e-function. The mathematical formalism is explained, using the example of the field of a decaying point source.

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ON THE THEORY OF THE ATMOSPHERIC ELECTRIC CURRENT FLOW, IV

INTRODUCTION

Problems in atmospheric electricity are usually treated according to the electrostatic theory in spite of the fact that a three-dimensional current flow is being dealt with here. The reason for this unsatisfactory state is that the electrostatic theory is worked out in great detail in numerous text books of mathematics and physics,^{1,2,3} but problems of the three-dimensional current-flow theory--especially with a variable conductivity or a convection current--are only briefly discussed, if at all. Therefore, the author started to publish some of his notes on the current-flow theory under the general title, "Zur Stroemungs-theorie des luftelektrischen Feldes I, II, III."^{4,5,6} Also, two papers,^{7,8} which deal with the calculation of the three-dimensional current flow under certain boundary conditions.

Preceding the author,⁸ Holzer and Saxon⁹ published their famous paper, "Distribution of Electrical Conduction Currents in the Vicinity of Thunderstorms," and in 1955 Tamura presented at the First International Conference on Atmospheric Electricity his excellent "Analysis of Electric Field after Lightning Discharges."¹⁰ In both papers the calculation was carried out according to the current-flow theory. But in the majority of publications in the field of atmospheric electricity, the electrostatic theory is still used for calculations, and this quite often leads to erroneous conclusions.

It soon became obvious that the solution of a few single problems was not enough to introduce the application of the theory of current flow to atmospheric electric problems in general. A more systematic discussion of the methods of solution, with a strong accent on the physical meaning of the mathematical manipulations and the consequences of the introduced assumptions, is necessary. This discussion is given in this report.

DISCUSSION

The Differential Equation of the Three-Dimensional Current Flow

Table I gives a list of the symbols used, with their meanings and dimensions.

In the electrostatic theory, only the quantities on the left side of the list are present; while in the current-flow theory there are, in addition, the four quantities of the right side. It is seen that the dimensions of the quantities of a corresponding pair differ only by the addition of the time unit, s , to the dimension of the electrostatic quantity. This will prove very fortunate in the conversion of solutions of the electrostatic theory into these of the corresponding solution of the current-flow theory, as will be shown later. However, this "fortunate" relationship has a deeper physical meaning, which becomes much clearer if the dimension of the charge cb (coulomb) instead of the current A (ampere) is introduced into the dimension of the quantities.

Table I. Symbols Used, with Their Meanings and Dimensions

$Q [As]$	= charge	$I [A]$	= strength of current source
$q \left[\frac{As}{m^3} \right]$	= space charge density	$\omega \left[\frac{A}{m^3} \right]$	= space current source density
$\vec{D} \left[\frac{As}{m^2} \right]$	= dielectric displacement	$\vec{i} \left[\frac{A}{m^2} \right]$	= current density
$\epsilon \left[\frac{As}{Vm} \right]$	= dielectric constant	$\lambda \left[\frac{A}{Vm} \right]$	= conductivity
$\vec{E} \left[\frac{V}{m} \right]$	= field strength		
$\phi [V]$	= potential function		

With the relation $As = cb$, the following are obtained:

$Q [cb]$	$I \left[\frac{cb}{s} \right]$
$q \left[\frac{cb}{m^3} \right]$	$\omega \left[\frac{cb}{sm^3} \right]$
$\vec{D} \left[\frac{cb}{m^2} \right]$	$\vec{i} \left[\frac{cb}{sm^2} \right]$
$\epsilon \left[\frac{cb}{Vm} \right]$	$\lambda \left[\frac{cb}{sVm} \right]$

In this way the time has been removed from the electrostatic quantities, where it does not belong, and introduced into the current-flow theory, where one would expect it to be. If the time is now extended, e.g., one second to infinity, then the movement of all particles would freeze and everything would be static. The four specific quantities of the current-flow theory, I , ω , \vec{i} , and λ , would vanish because the time is in the denominator of the dimensions, and the electrostatic conditions are obtained. So, one could say that electrostatic is a special case of the current-flow theory.

In the current-flow theory, all the quantities listed above can be functions of space and time. The problems encountered are mostly of the following type: Given are the spatial distribution and the time function of the current source ω and the electrical properties of space ϵ and λ as functions of space and time. This includes also the boundary condition such as a perfect conducting earth surface, or sometimes an ionosphere of infinite conductivity. The problem is to calculate the potential function and the field and current distribution.

The dielectric constant ϵ is always a true constant in space and time, since the difference of ϵ in air and in a vacuum is negligible. The manner in which the conductivity λ is given determines the grade of difficulty of the problem, and to some extent also the method of solution. Therefore, the problems may be subdivided into the following six classes:

1. The conductivity is constant in space and time.
2. The conductivity is constant in space, but a function of time.
3. The conductivity is constant in time, but a function of space.
4. The conductivity is a function of space and time.
5. The conductivity is a tensor (influence of the earth's magnetic field in the ionosphere).
6. The assumption for the definition of the conductivity is no longer valid.

This is the case in outer space, where the mean free path of the moving particle is greater than either the antenna of the measuring instrument or the space under consideration.

Classes 1, 2, and 3 are special cases of class 4. Class 4, again, could be considered as a special case of 5. However, the solution of 5 as well as of 6 class problems requires a much larger mathematical effort. Therefore, this report is limited to a general solution of 1 to 4 class problems.

The solution to class 1 problems can be obtained easily if the corresponding electrostatic solution is known. It is necessary only to substitute for the electrostatic quantities, Q , q , \vec{D} , and ϵ , the corresponding current-flow quantities I , w , \vec{I} , and λ of table I in the formulas of the potential function ϕ or of the electric field distribution \vec{E} . For instance, the potential function $\phi = \frac{Q}{4\pi\epsilon} \frac{1}{r}$ of a point charge leads to the potential function $\phi = \frac{I}{4\pi\lambda} \frac{1}{r}$ of a point current source.

The applicability of class 1 problems in atmospheric electricity seems to be very limited as, for instance, to the calculation of the effective altitude or area of field or current antennas, where the conductivity in the space considered may be assumed to be constant. But it is shown later that the electrostatic solution is needed in addition to that of the stationary current flow for the general solution of class 3 and 4 problems. Also, the important matching condition¹¹ for air-earth-current measurements can be obtained with the assumption that the conductivity is constant in space and time.

If the conductivity changes with time but is constant in space, this would result in mismatching, which is a problem belonging to class 2. An analysis of these conditions for air-earth-current measurement at the ground

is given by Ruhnke.¹² At a larger scale the same problem is encountered by the air-earth-current radiosonde,¹³ where the conductivity increases during the flight time from the low ground value by a factor 100 or more to that in the higher altitudes. Here the assumption is made that the conductivity can be considered as constant in the space occupied by the antennas of the radiosonde, but will increase in time with the ascent of the balloon.

The calculations given by the author^{5,7,13} belong to class 3.

There is no example of class 4 problems in the literature. This type of problem will be encountered by the calculation of the austausch generator, where the air turbulence will change the conductivity in space and time and also cause a convection current which is a function of space and time.

The differential equation of the three-dimensional current flow, which has to be fulfilled by all solutions, is derived from the second Maxwell equation and is well known as the equation of the continuous current flow. It is written in the following form:

$$\operatorname{div} \left(\frac{\partial \vec{D}}{\partial t} + \vec{i} \right) = \omega. \quad (1)$$

The intensity of the current production is usually given by a convection current $\vec{\zeta}$. In this case the space current source density ω is given by $\operatorname{div} \vec{\zeta}$. If this is introduced into Eq. (1),

$$\operatorname{div} \left(\frac{\partial \vec{D}}{\partial t} + \vec{i} \right) = \operatorname{div} \vec{\zeta}. \quad (2)$$

From Eq. (2) follows

$$\frac{\partial \vec{D}}{\partial t} + \vec{i} = \vec{\zeta} + \vec{c}. \quad (3)$$

\vec{c} , which appears here as a kind of integration constant with the condition $\operatorname{div} \vec{c} = 0$, is known as the complete Maxwell current. It is a current flow, which originates at the boundaries of the considered space. In thunderstorm problems, \vec{c} would represent the fair-weather current between the ionosphere and the earth, which penetrates the thunderstorm areas as well as the fair-weather areas. However, the fair-weather current density is much smaller than the current density caused by the thunderstorm, and therefore it is usually neglected. This cannot be done in the case of the austausch generator, which works in the fair-weather region. Here, the convection current density of the austausch generator $\vec{\zeta}$ and the current \vec{c} produced by the charged earth are of the same order of magnitude. (The conduction current density of \vec{c} is the well-known air-earth-current density.) Therefore, in this case, both current sources have to be considered. But as the principle of superposition also holds in the current-flow theory, the current distribution of the two sources may be calculated separately, and the solutions

superimposed. The dielectric displacement \vec{D} and the current density \vec{i} are connected with the electric field \vec{E} through the following two equations:

$$\epsilon \vec{E} = \vec{D}. \quad (4)$$

$$\lambda \vec{E} = \vec{i}. \quad (5)$$

(If λ is a tensor Λ as in class 5 problems, $\Lambda \cdot \vec{E} = \vec{i}$, and from here on the calculation of class 5 problems would branch off.) With Eqs. (4) and (5), \vec{E} is substituted for \vec{i} and \vec{D} in (3), and

$$\frac{\partial \vec{E}}{\partial t} + \frac{\lambda}{\epsilon} \vec{E} = \frac{1}{\epsilon} (\vec{c} + \vec{c}) \quad (6)$$

is obtained. The solution of this differential equation is

$$\vec{E} = e^{-\int \frac{\lambda dt}{\epsilon}} \left(\vec{E}(t=0) + \frac{1}{\epsilon} \int (\vec{c} + \vec{c}) e^{\int \frac{\lambda dt}{\epsilon}} dt \right). \quad (7)$$

This is the general solution for \vec{E} for all class 1 to 4 problems. The integration constant $\vec{E}(t=0)$ gives the field distribution for $t=0$. On account of the integration constant, the integral can be written without boundaries; and the letter t , which ordinarily would appear as the upper boundary of the integral, is used as the integration variable of the integral.

According to the above-mentioned principle of superposition, the field-vector \vec{E} may be split up into two parts; namely, \vec{E}_c , which results from the convection current \vec{c} , and \vec{E}_b , which results from current emitted at the boundaries.

$$\vec{E}_c = e^{-\int \frac{\lambda dt}{\epsilon}} \left(\vec{E}_c(t=0) + \frac{1}{\epsilon} \int \vec{c} e^{\int \frac{\lambda dt}{\epsilon}} dt \right). \quad (8)$$

and

$$\vec{E}_b = e^{-\int \frac{\lambda dt}{\epsilon}} \left(\vec{E}_b(t=0) + \frac{1}{\epsilon} \int \vec{c} e^{\int \frac{\lambda dt}{\epsilon}} dt \right). \quad (9)$$

Equation (8) cannot be discussed without further knowledge of the convection current \vec{c} . The analysis of Eq. (9), especially the physical meaning of the complete Maxwell current \vec{c} is given below.

The Solution of Boundary Problems without Convection Currents

Set $\vec{c} = 0$, and from Eq. (6) the electric field \vec{E}_b due to the boundary conditions may be obtained. With the further omission of the subscript b , it is

$$\frac{\partial \vec{E}}{\partial t} + \frac{\lambda}{\epsilon} \vec{E} = \frac{\vec{c}}{\epsilon}. \quad (10)$$

The form and the physical meaning of the complete Maxwell current \vec{c} will now be deduced from the three boundary values \vec{c}_b , \vec{c}_e , and \vec{c}_s which \vec{c} has to assume, first at the boundary (\vec{c}_b), second, if $\lambda = 0$ (\vec{c}_e), and third, if $\frac{\partial \vec{E}}{\partial t} = 0$ (\vec{c}_s).

At the boundary of the current source, the field E_b is given by the field strength F_b and the time function T of the current source. Therefore,

$$\vec{E}_b = \vec{F}_b \cdot T. \quad (11)$$

$\vec{F}_b(x,y,z)$ is hereby a function of the space coordinates x , y , and z only, and $T(t)$ is a pure time function. This implies that on all parts of the boundary of the current source the field changes with the same time function. From Eqs. (10) and (11) follows

$$\frac{\vec{c}_b}{\epsilon} = \vec{F}_b \frac{dT}{dt} + \frac{\lambda_b}{\epsilon} \vec{F}_b T. \quad (12)$$

If $\lambda = 0$, the electrostatic field distribution is obtained, which will be indicated by a subscript e at the Maxwell current \vec{c}_e and the field vector \vec{F}_e . Furthermore, the finite propagation velocity of the electric field with the speed of light will be neglected, which means that the electric field \vec{E} follows everywhere the time function T of the source momentarily. $\vec{E} = \vec{F}_e \cdot T$. Hereby excluded are all problems where the propagation velocity of electric signals becomes important, for instance, by the electromagnetic wave of a lightning discharge (sferics). From Eq. (10) follows

$$\frac{\vec{c}_e}{\epsilon} = \vec{F}_e \frac{dT}{dt}. \quad (13)$$

If the time function of the current source is a constant, the condition of the stationary current flow is obtained, which is indicated by the subscript s . With $\frac{\partial \vec{E}_s}{\partial t} = 0$ and $\vec{E}_s = \vec{F}_s T$, from Eq. (10) is obtained

$$\frac{\vec{c}_s}{\epsilon} = \frac{\lambda}{\epsilon} \vec{F}_s T. \quad (14)$$

Equations (12), (13), and (14) indicate the form which the Maxwell current \vec{c} in general will assume. It may be inferred that \vec{c} is composed in the following manner:

$$\vec{c} = \epsilon \vec{F}_e \frac{dT}{dt} + \lambda \vec{F}_s T. \quad (15)$$

Equation (15) fulfills all the required conditions. At the boundary, where $\vec{F}_{eb} = \vec{F}_{sb} = \vec{F}_b$, Eq. (15) changes to Eq. (12), and therefore meets the boundary conditions. Furthermore, $\text{div } \vec{c} = 0$, because $\text{div } \vec{F}_e = 0$ as well as $\text{div } \lambda \vec{F}_s = 0$. Therefore the complete Maxwell current \vec{c} is given in the form of Eq. (15).

If Eq. (15) is introduced into Eq. (9), with the omission of the subscript b, with $\vec{E}(t=0) = \vec{F}_0$,

$$\begin{aligned} \vec{E} = \vec{F}_0 \cdot e^{-\int \frac{\lambda dt}{\epsilon}} + \vec{F}_e \cdot e^{-\int \frac{\lambda dt}{\epsilon}} \int \frac{dT}{dt} e^{\int \frac{\lambda dt}{\epsilon}} dt \\ + \vec{F}_s \cdot e^{-\int \frac{\lambda dt}{\epsilon}} \int \frac{\lambda}{\epsilon} T e^{\int \frac{\lambda dt}{\epsilon}} dt. \end{aligned} \quad (16)$$

The physical meaning of Eq. (16) becomes much clearer if the following abbreviations are introduced for the time-function factors of \vec{F}_0 , \vec{F}_e , and \vec{F}_s . It will be

$$T_e = e^{-\int \frac{\lambda dt}{\epsilon}} \int \frac{dT}{dt} e^{\int \frac{\lambda dt}{\epsilon}} dt. \quad (17)$$

$$T_s = e^{-\int \frac{\lambda dt}{\epsilon}} \int \frac{\lambda}{\epsilon} T e^{\int \frac{\lambda dt}{\epsilon}} dt. \quad (18)$$

$$T_0 = e^{-\int \frac{\lambda dt}{\epsilon}}. \quad (19)$$

By partial integration of Eq. (17),

$$T_e = T - e^{-\int \frac{\lambda dt}{\epsilon}} \int \frac{\lambda}{\epsilon} T e^{\int \frac{\lambda dt}{\epsilon}}$$

is obtained, and with Eq. (18),

$$T_e = T - T_s, \quad T = T_e + T_s. \quad (20)$$

The sum of the time functions of the electrostatic field and the stationary current-flow field is the time function of the current source. These time-function factors are obtained by the prescribed integration process on the time function T of the source. They are different at each point in space, because λ is a function of space; and they are of different form for the electrostatic field \vec{F}_e , the stationary current-flow field \vec{F}_s , and the decaying initial field \vec{F}_0 .

If Eqs. (17), (18), and (19) are introduced into (16),

$$\vec{E} = \vec{F}_0 T_0 + \vec{F}_e T_e + \vec{F}_s T_s. \quad (21)$$

The electric field \vec{E} is composed of three vectors, each of them modulated by a different time function. \vec{F}_0 is the field distribution at the time $t = 0$ and decays with the time constant of the observation point. \vec{F}_e is the electrostatic field, and the weighting function T_e is zero for a constant-current source, but becomes very large for fast time variations of the source. \vec{F}_s is the field of the stationary current flow, and the factor T_s becomes one for a constant current source. It is pointed out that in general \vec{F}_0 , \vec{F}_e , and \vec{F}_s do not have the same directions, and the rules of vector addition have to be applied. After the decay of \vec{F}_0 , the field-vector \vec{E} is limited to the space between \vec{F}_e and \vec{F}_s .

To illustrate the point, the field lines of a point source are drawn in Fig. 1 for the electrostatic case (broken lines), and for the stationary current-flow case in which the conductivity increases with altitude according to an e-function (solid lines). It is seen that for almost any point in space the field direction and amplitude--the amplitude is indicated by the spacing of the lines--is different in the two cases. If one arbitrary observation point is selected, a picture of the composition of the field-vector \vec{E} is obtained, as given in Fig. 2. The weighting time functions will change with time, and accordingly the vector \vec{E} will change its amplitude and shift its position inside the hatched space.

For a fast time variation, \vec{E} is predominantly given by \vec{F}_e ; and for slow time variation, \vec{E} will move back to the position of \vec{F}_s . Hence it becomes obvious that for a complete solution the electrostatic field has to be calculated as well as the field of the stationary current flow. The electrostatic, the current flow, and the initial field \vec{F}_e , \vec{F}_s , and \vec{F}_0 are given by the gradient operation from their potential functions ϕ_e , ϕ_s , and ϕ_0 , respectively.

$$\vec{F}_e = - \text{grad } \phi_e, \vec{F}_s = - \text{grad } \phi_s, \vec{F}_0 = - \text{grad } \phi_0. \quad (22)$$

Hence, the electric field \vec{E} is also given by the gradient of a potential function ϕ . $\vec{E} = - \text{grad } \phi$. Therefore, with respect to Eq. (21),

$$\text{grad } \phi = T_0 \text{ grad } \phi_0 + T_e \text{ grad } \phi_e + T_s \text{ grad } \phi_s. \quad (23)$$

Equations (21) and (23) are the general solutions to all class 1 to 4 problems in the absence of a convection current.

Solution of the Differential Equation of the Stationary Current Flow

If the current flow is stationary, derivatives with respect to time are zero, and from Eq. (2) is obtained the differential equation of the stationary current flow.

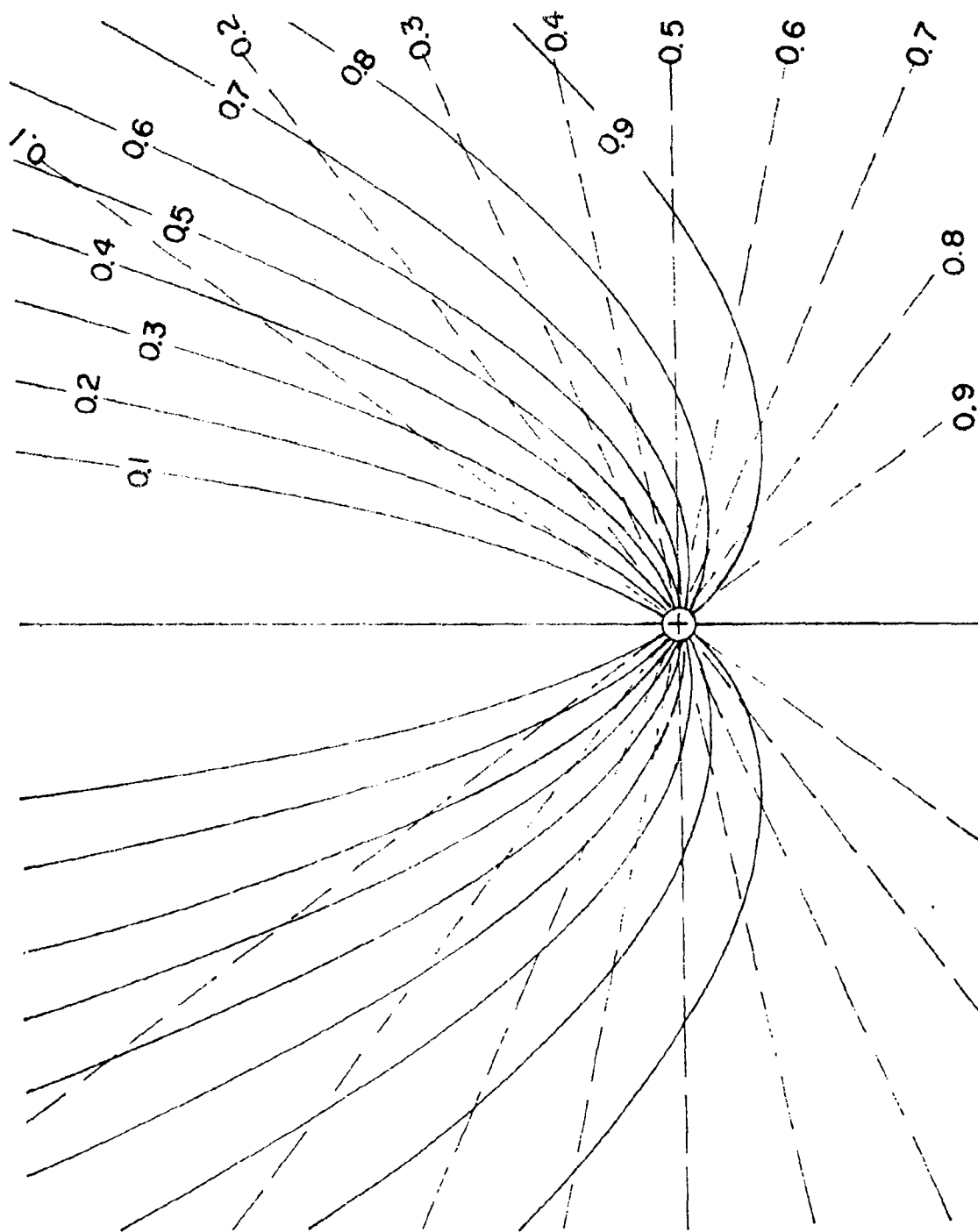


Fig. 1. Lines of the electric field or current flow of 1) a point source in a medium of constant conductivity (broken lines), and 2) a point source in a medium of conductivity, which increase with altitude (solid lines).

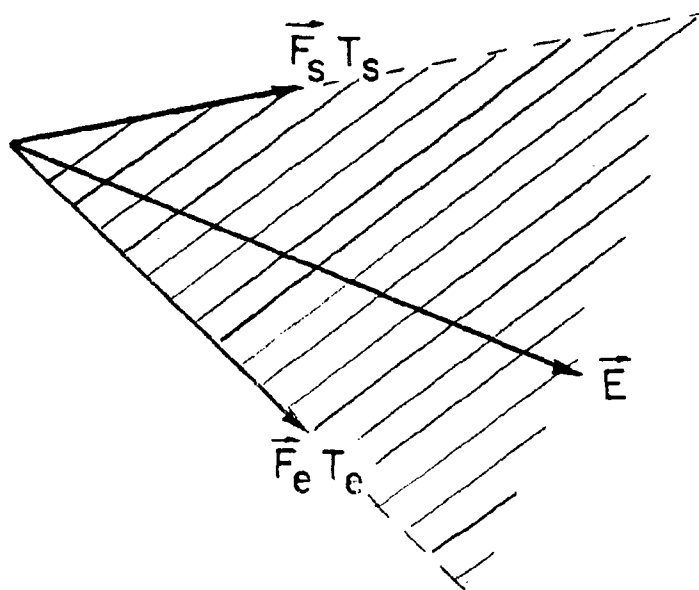


Fig. 2. Composition of the general field vector \vec{E} by vector addition from the electrostatic field vector \vec{F}_e weighted by the time function T_e and the stationary current flow field vector \vec{F}_s weighted by the time function T_s .

$$\operatorname{div} \vec{i}_s = 0. \quad (24)$$

With $\vec{i}_s = \lambda \vec{F}_s$ and $\vec{F}_s = -\operatorname{grad} \phi_s$,

$$\operatorname{div} \lambda \operatorname{grad} \phi_s = 0. \quad (25)$$

The following self-explanatory operations are carried out on this equation:

$$\operatorname{div} \lambda \operatorname{grad} \phi = 0.$$

$$\lambda \operatorname{div} \operatorname{grad} \phi + \operatorname{grad} \lambda \operatorname{grad} \phi = 0.$$

Divide by $\lambda^{\frac{1}{2}}$, add and subtract $\phi \operatorname{div} \operatorname{grad} \lambda^{\frac{1}{2}}$.

$$\lambda^{\frac{1}{2}} \operatorname{div} \operatorname{grad} \phi + 2 \operatorname{grad} \lambda^{\frac{1}{2}} \operatorname{grad} \phi + \phi \operatorname{div} \operatorname{grad} \lambda^{\frac{1}{2}} - \phi \operatorname{div} \operatorname{grad} \lambda^{\frac{1}{2}} = 0.$$

$$\operatorname{div} (\lambda^{\frac{1}{2}} \phi) - \phi \operatorname{div} \operatorname{grad} (\lambda^{\frac{1}{2}}) = 0.$$

$$\frac{\operatorname{div} \operatorname{grad} \lambda^{\frac{1}{2}} \phi}{\lambda^{\frac{1}{2}} \phi} - \frac{\operatorname{div} \operatorname{grad} \lambda^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}} = 0. \quad (26)$$

Here are defined two new functions:

$$M = \lambda^{\frac{1}{2}} \phi \text{ and } N = \lambda^{\frac{1}{2}}. \quad (27)$$

Introducing these new functions into Eq. (24), the relatively simple equation is obtained:

$$\frac{\operatorname{div} \operatorname{grad} M}{M} = \frac{\operatorname{div} \operatorname{grad} N}{N}. \quad (28)$$

With this equation there are available all the solutions of the electrostatic theory $\operatorname{div} \operatorname{grad} M = 0$, if it is possible to present λ by a function N^2 , of which $\operatorname{div} \operatorname{grad} N = 0$.

It is pointed out that different coordinate systems may be used for the representation of M and N because the operation $\operatorname{div} \operatorname{grad}$ is invariant against a change of coordinates.

As an example, assume that it is desired to calculate the potential function of a point source imbedded in a medium where the conductivity can be represented or approximated in the space under consideration by a suitable piece of a parabola.

$$\lambda = \lambda_0 (mz - a)^2. \quad (29)$$

The parameters m , a , and λ_0 can be used to select the best-fitting piece of the parabola.

From Eq. (27) follows

$$N = \lambda^{\frac{1}{2}} = \lambda_0^{\frac{1}{2}} (mz - a), \quad (30)$$

and

$$\text{div grad } N = \frac{d^2 N}{dz^2} = 0. \quad (31)$$

Introducing this result in Eq. (28),

$$\text{div grad } M = 0. \quad (32)$$

This equation means that any known electrostatic potential function or any sum of them may be chosen in any coordinate system which suits the purpose best. For a point source, polar coordinates would certainly be chosen, where the function M is given by

$$M = \frac{A}{r}. \quad (33)$$

A is here a constant given by the strength of the source I , and r is the distance of the observation point from the center of the point source. If the point source is located in the cartesian coordinates at the point x_0 ,

y_0, z_0 , then r is given by $r = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{1}{2}}$. The potential function is now easily obtained by Eqs. (27), (29), and (33).

$$\phi = \frac{M}{\lambda^{\frac{1}{2}}} = \frac{A}{\lambda^{\frac{1}{2}} (mz - a) r}. \quad (34)$$

The determination of A remains as the last step. If the point source is inclosed by a small sphere of the radius r_0 , which is chosen so small that the conductivity λ can be considered as constant with the value at the center of the sphere $\lambda = \lambda_0 (m z_0 - a)^2$, then, according to Eq. (34),

$$\phi = \frac{A}{\lambda_0^{\frac{1}{2}} (m z_0 - a) r}. \quad (35)$$

The field in the r direction on the little sphere $r = r_0$ is given by

$$F = - \frac{\partial \phi}{\partial r} = \frac{A}{\lambda_0^{\frac{1}{2}} (m z_0 - a)} \frac{1}{r_0^2},$$

and the current density according to Ohm's law $i = \lambda F$,

$$i_r = \frac{A \lambda_0 (m z_0 - a)^2}{\lambda_0^{\frac{1}{2}} (m z_0 - a) r_0^2} = \frac{A \lambda_0^{\frac{1}{2}} (m z_0 - a)}{r_0^2}.$$

Integration over the surface O of the little sphere gives the current strength I of the source.

$$I = \int_O i_r dO = \frac{A \lambda_0^{\frac{1}{2}} (m z_0 - a)}{r_0^2} 4 \pi r_0^2,$$

or

$$A = \frac{I}{4 \pi \lambda_0^{\frac{1}{2}} (m z_0 - a)} \quad (36)$$

If this result is introduced in Eq. (34), the final solution for the potential function ϕ_s of the problem is obtained.

$$\phi_s = \frac{1}{4 \pi \lambda_0 (m z_0 - a)} \cdot \frac{1}{(m z - a) \cdot r} \quad (37)$$

Unfortunately, in atmospheric electric problems, the conductivity is usually represented by an e-function. For instance, in a cartesian coordinate system x, y, z , in the manner

$$\lambda = \lambda_0 e^{2 k z} \quad (38)$$

The x, y plane with $z = 0$ represents hereby the earth's surface, and λ_0 the conductivity at the ground. This function (given in Eq. (38)) does not fulfill Laplace's equation, but it is

$$\text{div grad } \lambda^{\frac{1}{2}} = \lambda_0^{\frac{1}{2}} k^2 e^{k z} = k^2 N, \text{ and}$$

$$\frac{\text{div grad } N}{N} = k^2 \quad (39)$$

Therefore, the function M is given as the solution of the differential equation

$$\text{div grad } M - k^2 M = 0 \quad (40)$$

To solve Eq. (40), the method of the eigen functions will be applied, but confined to problems of rotational symmetry. This means that, for instance, in polar coordinates r, θ, ϕ , the function M is given by the product of a function $f(r)$, which depends on r only, and another function $g(\theta)$, which depends only on θ .

$$M = f(r) g(\theta) \quad (41)$$

This leads with Eq. (40) to the differential equation

$$\text{div grad } f \cdot g - k^2 f \cdot g = 0,$$

or, in polar coordinates,

$$g \left(\frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} \right) + \frac{f}{r^2 \sin \theta} \frac{d}{d\theta} \sin \theta \frac{dg}{d\theta} - k^2 f g = 0, \quad (42)$$

or

$$\frac{1}{k^2 f} \left(\frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} \right) + \frac{1}{r^2 k^2 g \sin \theta} \frac{d}{d\theta} \sin \theta \frac{dg}{d\theta} - 1 = 0. \quad (43)$$

$$\text{If } \frac{1}{g \sin \theta} \frac{d}{d\theta} \sin \theta \frac{dg}{d\theta} = -n(n+1) \quad (44)$$

where $n = 0, 1, 2, 3, \dots$, the spherical functions of the first kind P_n (Legendre's functions) are obtained as a solution for g .

$$g = P_n(\cos \theta). \quad (45)$$

If Eq. (44) is inserted in (43), the differential equation

$$\frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} - \left(\frac{n(n+1)}{r^2} + k^2 \right) f = 0 \text{ is obtained for } f. \quad (46)$$

Equation 46 can be solved with the help of cylinder functions Z_p , with a noninteger parameter $p = n + \frac{1}{2}$. The solution is given by

$$f = (j k r)^{-\frac{1}{2}} Z_{n+\frac{1}{2}}(j k r). \quad (47)$$

Thereby, j is the imaginary unit $j = \sqrt{-1}$.

The complete solution for M follows from Eqs. (41), (45), and (47).

$$M_n = A_n P_n (j k r)^{-\frac{1}{2}} Z_{n+\frac{1}{2}}(j k r). \quad (48)$$

A_n is an arbitrary constant, which serves to meet the prescribed boundary condition. M may be presented in a more general way as a superposition of the M_n .

$$M = \sum_{n=0}^{\infty} M_n. \quad (49)$$

The cylinder functions Z_p of a noninteger parameter p can be expressed in given polynomials of the argument and sin and cos functions. For instance, for $n = 0$,

$$Z_{\frac{1}{2}} = \left(\frac{2}{\pi j k r} \right)^{\frac{1}{2}} \sin j k r.$$

$$\text{With } P_0 = 1, \text{ and } \sin j k r = j \sinh k r = \frac{j}{2} [\exp(kr) - \exp(-kr)],$$

$$M_0 = A_0 \left[\frac{\exp(kr)}{kr} - \frac{\exp(-kr)}{kr} \right]. \quad (50)$$

The constant factor $(2)^{-\frac{1}{2}}$ is hereby combined with A_0 .

Each of the e-function terms in the bracket of Eq. (50) fulfills the differential equation. Hence, if the boundary condition requests a pole ($M_0 \rightarrow \infty$) at the origin $r = 0$, only the second term is used. This will then lead to the potential function of a point source.

According to Eq. (27), the potential function is given by

$$\phi = M \lambda^{-\frac{1}{2}}.$$

Using only the second term in the bracket of Eq. (50), and letting A_0 take care of the sign and the constant factor $\lambda_0^{-\frac{1}{2}}$, the potential function of a point source is obtained.

$$\phi = A_0 \frac{\exp [-k(r + z)]}{kr} . \quad (51)$$

In a manner very similar as before, it is possible to calculate from Eq. (51) the field strength in the r direction on the surface of a very small sphere around the current source I , convert the field into the current density, and integrate over the surface. This gives the strength of the current source I , and the constant A_0 can be determined. It is

$$A_0 = \frac{I k}{4 \pi \lambda_0} , \quad (52)$$

whereby λ_0 denotes the conductivity value at the current source. From Eqs. (51) and (52), the final solution is obtained:

$$\phi = \frac{I}{4 \pi \lambda_0} \frac{\exp [-k(r + z)]}{r} . \quad (53)$$

For $k = 0$, Eq. (38) is $\lambda = \lambda_0 \exp (2 k z) = \lambda_0$, a constant, and Eq. (53) simplifies to

$$\phi = \frac{I}{4 \pi \lambda_0} \cdot \frac{1}{r} .$$

This is the right solution for the point source in a medium of constant conductivity.

The equation (53), which follows here as the simplest solution of Eqs. (48) and (49), is the key solution in the calculation of the electric field of thunderstorms of Holzer and Saxon,⁹ in the calculation of the recovery curve of lightning discharges by Tamura,¹⁰ and in the calculation of the thunderstorm generator by the author.⁸

It is mentioned here that a similar procedure to that used for polar coordinates led to the eigen functions of the differential equation, $\text{div grad } M - k^2 M = 0$ in cylindrical coordinates z, R, ϕ . Again assuming rotational symmetry,

$$M_n = A_n Z_0(n k R) \exp \left[\pm (1 - n^2)^{\frac{1}{2}} k z \right] . \quad (54)$$

Z_0 is the cylinder function of the order zero. n is here not confined to an integer, but may assume any value. For $n = 0$,

$$M_0 = A_0 \exp (\pm k z) .$$

It is seen that the root of the conductivity function $\lambda^{\frac{1}{2}} = \lambda_0^{\frac{1}{2}} e^{kz}$ is the eigen function of the order zero in cylindrical coordinates. In cartesian coordinates the eigen functions are given by

$$M_n = A_n \exp(\pm c_1 x) \exp(\pm c_2 y) \exp(\pm c_3 z). \quad (55)$$

c_1 , c_2 , and c_3 are arbitrary functions of n , restricted only by the equation

$$c_1^2 + c_2^2 + c_3^2 = k^2. \quad (56)$$

A final remark is made about the representation of λ in polar coordinates. It is seen from Eq. (48) and from Eq. (50) that a pure e -function is not an eigen function in the polar coordinate system. Therefore it is not possible to obtain a simple solution of the potential function ϕ for $\lambda = \lambda_0 \exp(2kr)$. The best that can be done is to choose the following function for λ :

$$\lambda = \lambda_0 \frac{r_0^2 \exp[2k(r - r_0)]}{r^2}. \quad (57)$$

In this case,

$$N = \lambda^{\frac{1}{2}} = \lambda_0^{\frac{1}{2}} \exp(-k r_0) \cdot r_0 \cdot \frac{e^{kr}}{r},$$

which is an eigen function according to Eq. (50). A_0 is given by $A_0 = \lambda^{\frac{1}{2}} r_0 \exp(-k r_0)$. If r_0 denotes the earth's radius, and this discussion is confined to the space between the earth's surface and about 100-km altitude, then the r^2 in the denominator changes very little, but the e -function in the nominator increases in the requested way. Hence, the representation of λ by Eq. (57) is absolutely feasible. This will, then, bring again the full benefit of the simple solutions as outlined above.

The Mirror Law in the Current-Flow Theory

To introduce the earth's surface as an equipotential layer, the mirror law is applied in the electrostatic theory. As this is a powerful method which leads to simple solutions of boundary problems, its application in the current-flow theory will be discussed briefly.

Take a point source I and place it in a cylindrical coordinate system z, R, ϕ on the positive z axis at the point $z = h$. With regard to the coordinate system used for formula (53), the zero point has been shifted down the z axis by the distance h . As can be easily verified, this means that the potential function ϕ is now expressed in the following form:

$$\phi_n = \frac{I}{4\pi\lambda_n} \frac{\exp[-k(r + z - h)]}{r}. \quad (58)$$

$\lambda_n = \lambda_0 \exp(2kh)$ is here the conductivity at the altitude h , and r is given by $r = [R^2 + (z - h)^2]^{\frac{1}{2}}$.

To now introduce the plane $z = 0$ (earth surface) as an equipotential layer, place, analog to the electrostatic theory, a point source of the strength I^* at the mirror point of I , i.e., at $z = -h$. The potential function ϕ^* of this source will be

$$\phi^* = \frac{I^*}{4 \pi \lambda^*} \frac{\exp [-k(r^* + z + h)]}{r^*}, \quad (59)$$

with $r^* = [(z + h)^2 + R^2]^{\frac{1}{2}}$ and $\lambda^* = \lambda_0 \exp (-2k h)$, the conductivity value at the point $z = -h$.

The superposition of ϕ_h and ϕ^* results in a potential function ϕ , which will be zero for $z = 0$.

$$\begin{aligned} \phi = \phi_h + \phi^* = \frac{I}{4 \pi \lambda_h} \frac{\exp [-k(r + z - h)]}{r} \\ + \frac{I^*}{4 \pi \lambda^*} \frac{\exp [-k(r^* + z + h)]}{r^*}. \end{aligned} \quad (60)$$

For $z = 0$ is $r = r^*$, $0 = \frac{I}{\lambda_h} \exp (kh) + \frac{I^*}{\lambda^*} \exp (-kh)$, or

$$I^* = - \frac{\lambda^* \exp (2 kh)}{\lambda_h} I. \quad (61)$$

If Eq. (61) is inserted in Eq. (60), the final formula for the potential function ϕ is obtained:

$$\phi = \frac{I}{4 \pi \lambda_h} \left[\frac{\exp (-kr)}{r} - \frac{\exp (-kr^*)}{r^*} \right] \exp -k(z-h). \quad (62)$$

For $k = 0$, Eq. (62) will change to the potential function ϕ_e of the electrostatic theory.

$$\phi_e = \frac{I}{4 \pi \lambda_h} \left[\frac{1}{r} - \frac{1}{r^*} \right]. \quad (63)$$

With $\lambda^* \exp (2 kh) = \lambda_0$, from Eq. (61),

$$I^* = - \frac{\lambda_0}{\lambda_h} I. \quad (64)$$

In the electrostatic theory, the mirror source is placed at the mirror point $-h$ and its strength is of the same amount but of opposite sign of the original source. It is seen here that in the current-flow theory the location and the sign reversal are retained, but that the amount of the mirror source is smaller than the original source in the same proportion as the

conductivity at the ground λ_0 is smaller compared to the conductivity λ_h at the height of the original source.

Applied to the current flow of a thunderstorm, this result has far-reaching consequences, as pointed out by the author.⁸ It means that only a part of the conduction current produced by the thunderstorm--namely, that given by I^* --flows to the earth's surface in the immediate neighborhood of the storm, while the other part is drained off to the ionosphere and contributes to the air-earth current of the fair-weather areas. The difference between the lines of current flow of a point source in a medium of constant conductivity, $\lambda = \lambda_0$ (equivalent to the field lines of the electrostatic case), and those of a point source in a medium of increasing conductivity, $\lambda = \lambda_0 \exp(2kz)$, can be easily recognized from Figs. 3a and 3b.⁸ This difference also affects very drastically the amount of charges of the thunderstorm, as calculated from field measurements at the ground,^{14, 15} the field reversal at the ground from a bipolar thunderstorm, the recovery curve of the lightning stroke,¹⁰ and many other phenomena. But, as the object of this report is to outline the methods of calculation more than to discuss specific application, the mathematical aspects will be continued.

The Potential Function of a Decaying Current Source

This problem finds its application in the calculation of the recovery curve after a lightning discharge. Field records of a thunderstorm taken at the ground show that each lightning flash increases (or decreases) the so-called stationary field of the thunderstorm very suddenly, and that after the flash is completed the field returns to its preflash value in an e-function fashion, whereby the time constant of the decay is approximately ten seconds. The return is called the recovery curve.

This phenomenon is rather puzzling because the time constant of the air at the ground is about 600 seconds, and charges separated or produced by the lightning flash should therefore decay much slower. Tamura¹⁰ showed in an excellent analysis that a much faster decrease of the field at the ground will result if, besides the decay of the lightning charges with the time constant in the cloud and the regenerating effect of the thunderstorm's charging mechanism, the screening effect of the space charge is taken into account, which builds up under the influence of the conductivity which increases with altitude. In other words, his calculations are based on the theory of current flow. The calculation given here may be considered as a part of Tamura's analysis, but presented from a different point of view, namely, to show a classical example of the full application of the current-flow theory.

The key problem is that of the decaying point charge. Assume that the charge $+Q_0$ is deposited by the lightning flash at a point at the altitude h . (If it is a cloud flash, in addition a negative charge $-Q_0$ would have to be placed at a higher altitude H and the resulting fields superimposed.) The charge decays with the time constant θ' at this altitude. $\theta' = \frac{e}{\lambda'}$. The charge Q left at the time t is then given by

$$Q = Q_0 e^{-\frac{\lambda' t}{e}} \quad (65)$$

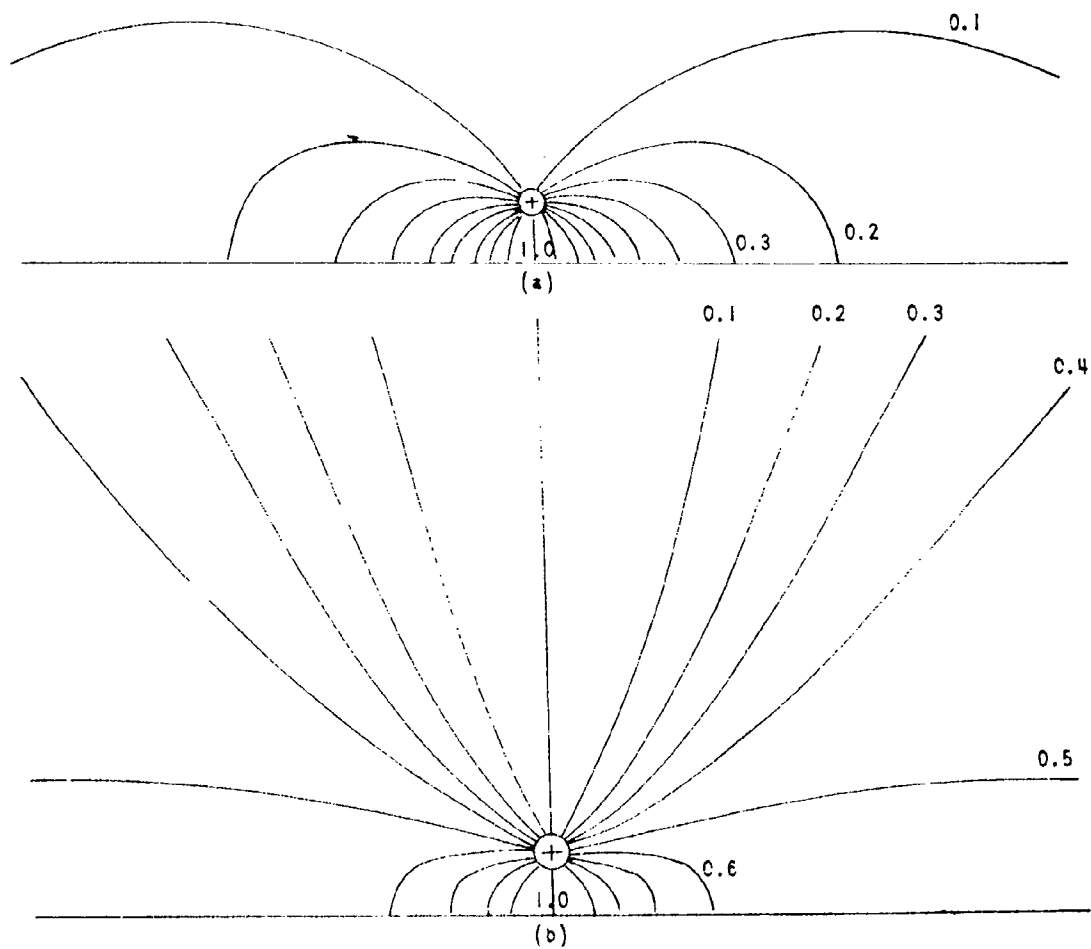


FIG. 3. FIELD AND CURRENT LINES OF A POINT SOURCE ABOVE A CONDUCTING PLANE IN A MEDIUM WITH
 (a) CONSTANT CONDUCTIVITY
 (b) CONDUCTIVITY INCREASING WITH ALTITUDE

The current output I is given by

$$I = - \frac{dQ}{dt} = \frac{\lambda'}{\epsilon} Q_0 e^{-\frac{\lambda' t}{\epsilon}}. \quad (66)$$

Therefore, the time function T of the current source is given by the exponential function of Eq. (66).

$$T = e^{-\frac{\lambda' t}{\epsilon}}. \quad (67)$$

The conductivity λ depends here only on the space coordinates, but is constant with time. The weighting time functions T_e , T_s , and T_c given by Eqs. (17), (18), and (19) simplify to

$$T_e = e^{-\frac{\lambda t}{\epsilon}} \int \frac{dT}{dt} e^{\frac{\lambda t}{\epsilon}} dt. \quad (68)$$

$$T_s = e^{-\frac{\lambda t}{\epsilon}} \int T e^{\frac{\lambda t}{\epsilon}} dt. \quad (69)$$

$$T_o = e^{-\frac{\lambda t}{\epsilon}}. \quad (70)$$

With T given by Eq. (67), the integrals can be solved, and it becomes

$$T_e = - \frac{\lambda'}{\lambda - \lambda'} T. \quad (71)$$

$$T_s = \frac{\lambda}{\lambda - \lambda'} T. \quad (72)$$

The equation $T_e + T_s = T$ is fulfilled by Eqs. (71) and (72). The next step is to determine \vec{F}_o from the initial condition. For $t = 0$, it is found that the field \vec{F} shall be the electrostatic field \vec{F}_e , since the lightning flash occurs in such a short time that the space charges of the stationary current-flow field due to the conductivity variation have not accumulated. From Eqs. (21), (67), (70), (71), and (72) follows for $t = 0$,

$$\vec{F}_e = \vec{F}_o - \frac{\lambda'}{\lambda - \lambda'} \vec{F}_e + \frac{\lambda}{\lambda - \lambda'} \vec{F}_s,$$

or

$$\vec{F}_o = \frac{\lambda}{\lambda - \lambda'} \vec{F}_e + \frac{\lambda}{\lambda - \lambda'} \vec{F}_s. \quad (73)$$

If Eq. (73) is introduced into Eq. (21), obtained with regard to Eqs. (71) and (72) is

$$\vec{E} = \frac{\lambda}{\lambda - \lambda'} (T_0 - \frac{\lambda'}{\lambda} T) \vec{F}_e + \frac{\lambda}{\lambda - \lambda'} (T - T_0) \vec{F}_s. \quad (74)$$

Equation (74) is the final form of the field equation of the decaying point source. It is valid for every point in space. But it must be remembered that vector addition is required, because \vec{F}_e and \vec{F}_s do not have the same direction. Only at the ground is the direction of \vec{E} , \vec{F}_e , and \vec{F}_s the same, namely, vertical to the ground surface. In this case the vector equation (74) simplifies to a scalar equation. Notice that the sum of the weighting time functions of \vec{F}_e and \vec{F}_s again is T , the time function of the current source.

REFERENCES

1. Courant, R., and D. Hilbert, *Methoden der Mathematischen Physik I, II*, Verlag Julius Springer, Berlin, 1937.
2. Smyth, W. R., *Static and Dynamic Electricity*, McGraw-Hill Book Co., New York, 1950.
3. Ollendorff, F., *Potentialfelder der Elektrotechnik*, Verlag Julius Springer, Berlin, 1932.
4. Kasemir, H. W., *Zur Stroemungstheorie des Luftelektrischen Feldes I*, Arch. Met. Geophys. Biokl., A 3, 85-97, 1950.
5. Kasemir, H. W., *Zur Stroemungstheorie des Luftelektrischen Feldes II*, Arch. Met. Geophys. Biokl., A 5, 56-70, 1952.
6. Kasemir, H. W., *Zur Stroemungstheorie des Luftelektrischen Feldes III*, Arch. Met. Geophys. Biokl., A 9, 357-370, 1956.
7. Kasemir, H. W., *Die Stromausbeute des Gewittergenerators*, *Ber. d. Wetterd. d. US Zone* 6, Nr. 38, 428-434, 1952.
8. Kasemir, H. W., *Der Gewittergenerator im Luftelektrischen Stromkreis I, II*, Z. f. Geophys., 25, 33-96, 1959.
9. Holzer, R. E., and D. S. Saxon, *Distribution of Electrical Conduction Currents in the Vicinity of Thunderstorms*, J. Geophys. Res., 57, 207-216, 1952.
10. Tamura, Y., *An Analysis of Electric Field after Lightning Discharges*, Proc. Conf. Atmospheric Electricity, 207-216, 1955.
11. Kasemir, H. W., *Measurement of the Air-Earth Current Density*, Proc. Conf. Atmospheric Electricity, 91-95, 1955.
12. Ruhnke, L. H., *The Effect of Mismatching in the Measurement of the Air-Earth Current Density*, USASRD Technical Report 2232, Fort Monmouth, N. J., Oct 61.

13. Kasemir, H. W., A Radiosonde for Measuring the Air-Earth Current Density, USASRDL Technical Report 2125, Fort Monmouth, N. J., Jun 60.
14. Simpson, G. C., and G. D. Robinson, The Distribution of Electricity in Thunderclouds, Proc. Roy. Soc., A 177, 281-329, 1940
15. Malan, D. J., Les Descharges dans l Air et la Charge Inference Positive d'un Nuage Orageux, Ann. Geophys., 8, 385-401, 1952.

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<p>1. Current flow 2. Atmospheric electricity</p>	<p>I. Kasemir, Heinz W. II. Army Electronics Research and Development Laboratories, Fort Monmouth, N. J.</p>	<p>1. Current flow 2. Atmospheric electricity</p>	<p>I. Kasemir, Heinz W. II. Army Electronics Research and Development Laboratories, Fort Monmouth, N. J.</p>
<p>ON THE THEORY OF THE ATMOSPHERIC ELECTRIC CURRENT FLOW, IV, by Heinz W. Kasemir, October 1963, 22 p. incl. illus. (USAEIRL Technical Report 2394) (DA Task LAO-11001-3-021-03)</p>	<p>Starting from the differential equation of the continuity of the current flow, a general solution is given for problems in which the conductivity is given as a function of space and time. A physical interpretation for the complete Maxwell current is obtained, and it is shown how the Maxwell current can be composed of two field vectors, namely, that of the electrostatic field and of the stationary current flow. Each vector is weighted by a different time function, which can be calculated from the time function of the current source.</p>	<p>ON THE THEORY OF THE ATMOSPHERIC ELECTRIC CURRENT FLOW, IV, by Heinz W. Kasemir, October 1963, 22 p. incl. illus. (USAEIRL Technical Report 2394) (DA Task LAO-11001-3-021-03)</p>	<p>III. DA Task LAO-11001-3-021-03</p>
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